

SOME TOPOLOGICAL OBSTRUCTIONS TO BOCHNER-KAEHLER METRICS AND THEIR APPLICATIONS

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1. Introduction

Let M^n be a compact (complex) manifold of complex dimension n . Let L be a line bundle over M^n . Denote by $H^i(M^n, L)$ the i -th cohomology group with coefficients in the sheaf of germs of local holomorphic sections in L , and by K and 1 the canonical line bundle and the trivial line bundle over M^n respectively. The m -genus or the plurigenera of M^n are given by

$$P_m = \dim H^0(M^n, mK) .$$

where $mK = K \otimes \dots \otimes K$ (m copies). P_1 is also called the geometric genus p_g of M . By the Serre duality theorem:

$$H^i(M^n, L) \cong H^{n-i}(M^n, L^{-1} \otimes K) ,$$

we also have $p_g = \dim H^n(M^n, 1)$. Put

$$g_i = \dim H^i(M^n, 1) .$$

Then g_1 is called the irregularity of M^n , denoted by q . The arithmetic genus is then given by

$$\alpha = 1 - g_1 + g_2 - \dots + (-1)^n g_n .$$

In particular, if M^n is a surface (we call a compact connected complex surface free from singularities simply a *surface*), $\alpha = 1 - q + p_g$. It is well-known that α, q, P_m are birational invariants.

In the following we denote by τ, χ, b_i and c_i the Hirzebruch signature, the Euler characteristic, the i -th Betti number and the i -th Chern class of M^n respectively. Let $c \in H^{2n}(M^n, \mathbb{Z})$ be a $2n$ -th cohomology class of M^n . We shall also regard c as the integer obtained from the cohomology class c by taking its value on the fundamental cyclic of M^{2n} .

Let g be a Kaehler metric on M^n . We denote by R_{jkl}^i, R_{ij} and ρ respectively

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the components of the curvature tensor R , the Ricci tensor S and the scalar curvature of (M^n, g) . In [2] (see also [14]), S. Bochner introduced a tensor field by

$$B_{j\bar{k}l}^i = R_{j\bar{k}l}^i - \frac{1}{2(n+2)}(R_{ik}\delta_{jl} + R_{i\bar{j}}\delta_{kl} + \delta_{ik}R_{j\bar{l}} + \delta_{i\bar{j}}R_{kl}) \\ + \frac{\rho}{4(n+1)(n+2)}(\delta_{ik}\delta_{jl} + \delta_{i\bar{j}}\delta_{kl}),$$

where δ_{ij} denote the Kronecker deltas. This tensor is considered as a complex version of the Weyl conformal curvature tensor and is called the Bochner curvature tensor. A Kaehler metric g on M^n is called a *Bochner-Kaehler metric* if its Bochner curvature tensor vanishes.

It is a basic problem in geometry to determine the class of surfaces or manifolds which do or do not admit Bochner-Kaehler metric.

In § 3 we shall give some obstructions to Bochner-Kaehler metric in terms of Hirzebruch signature, Euler characteristic and arithmetic genus. In § 4, we shall show most surfaces admit no Bochner-Kaehler metric. In § 5, we shall introduce some special (algebraic) surfaces which we shall use in the later sections and we shall mention some results of Šafarevič. In § 6, we shall give classification theorems for Bochner-Kaehler surfaces for analytic case.

2. Preliminaries

Let M^n be an n -dimensional complex manifold with a Kaehler metric g , and $\theta^1, \dots, \theta^n$ a field of unitary coframes. Then

$$g = \frac{1}{2} \sum (\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i).$$

Here and in § 3 we use the ranges $i, j, k, l, \dots = 1, \dots, n$. The fundamental 2-form is given by

$$\Phi = \frac{\sqrt{-1}}{2} \sum \theta^i \wedge \bar{\theta}^i,$$

which is a harmonic form. Let

$$\Omega_j^i = \sum R_{j\bar{k}l}^i \theta^k \wedge \bar{\theta}^l$$

be the curvature form of M . Then the curvature tensor of M is the tensor field with local components $R_{j\bar{k}l}^i$, which will be denoted by R . The Ricci tensor S and the scalar curvature ρ are given respectively by

$$S = \frac{1}{2} \sum (R_{i\bar{j}}\theta^j \otimes \bar{\theta}^i + \bar{R}_{i\bar{j}}\bar{\theta}^j \otimes \theta^i), \quad \rho = 2 \sum R_{i\bar{i}},$$

where $R_{i\bar{j}} = 2 \sum R_{ik\bar{j}}$. We denote by $\|R\|$, $\|S\|$ and $\|B\|$ the length of the curvature tensor, the Ricci tensor and the Bochner curvature tensor respectively, so that

$$\begin{aligned}\|R\|^2 &= 16 \sum R_{jk\bar{l}}^i R_{i\bar{l}k}^j, & \|S\|^2 &= 2 \sum R_{i\bar{j}} R_{j\bar{i}}, \\ \|B\|^2 &= 16 \sum B_{jk\bar{l}}^i B_{i\bar{l}k}^j.\end{aligned}$$

It is easily seen that

$$(1) \quad \|B\|^2 = \|R\|^2 - \frac{8}{n+2} \|S\|^2 + \frac{2}{(n+1)(n+2)} \rho^2.$$

A Kaehler manifold is called a *space form* with constant holomorphic sectional curvature c if we have

$$R_{jk\bar{l}}^i = \frac{c}{4} (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}).$$

A space form with vanishing holomorphic sectional curvature is called a flat manifold.

We state the following general lemma for later use.

Lemma 1. *Let M be an n -dimensional Kaehler manifold. Then*

$$\frac{1}{2} n(n+1) \|R\|^2 \geq 2n \|S\|^2 \geq \rho^2.$$

The first equality holds if and only if M is a space form, and the second equality holds if and only if M is Einsteinian.

Proof. The first inequality is obtained by considering the length of the tensor field with components

$$R_{jk\bar{l}}^i - \frac{1}{2(n+1)} (R_{j\bar{k}} \delta_{il} + R_{k\bar{l}} \delta_{ij}).$$

It is well-known that this tensor field vanishes if and only if M is a space form. The second inequality is obtained by considering the tensor field with components

$$R_{i\bar{j}} - \frac{\rho}{2n} \delta_{ij}.$$

3. Topological obstructions

In this section we shall give some obstructions to Bochner-Kaehler metric and Einstein-Kaehler metric for surfaces.

Proposition 1 [3]. *If a surface M admits a Bochner-Kaehler metric g , then we have the following inequalities:*

- (i) $\tau \geq 0$;
- (ii) $\chi \leq 4\alpha$;
- (iii) $\chi \leq 3\tau$;
- (iv) $\alpha \leq \tau$.

Equality of (i) or (ii) holds if and only if (M, g) is either flat or locally a product surface of two curves, one with constant positive Gauss curvature H and the other with constant negative Gauss curvature $-H$.

Equality of (iii) or (iv) holds if and only if (M, g) is a space form.

Proof. Let M be a surface. Then the first and second Chern classes c_1 and c_2 are represented by

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi} \sum \Omega_i^i,$$

and

$$\gamma_2 = -\frac{1}{8\pi^2} \sum (\Omega_i^i \wedge \Omega_j^j - \Omega_j^j \wedge \Omega_i^i),$$

respectively. The first Pontriagin class p_1 is given by

$$(2) \quad p_1 = c_1^2 - 2c_2.$$

From the Hirzebruch signature theorem, the Riemann-Roch-Hirzebruch theorem and the Gauss-Bonnet-Chern theorem [1], [6], [10], we have

$$(3) \quad \tau = \frac{1}{3} \int_M * \tilde{p}_1,$$

$$(4) \quad \alpha = \frac{1}{12} \int_M * (\gamma_1^2 + \gamma_2),$$

$$(5) \quad \chi = \int_M * \gamma_2,$$

where $*$ is the Hodge star operator, and \tilde{p}_1 the first Pontriagin form. By straightforward computations, we may find that

$$(6) \quad \tau = \frac{-1}{2^4 \cdot 3 \cdot \pi^2} \int_M (\|R\|^2 - 2\|S\|^2) * 1,$$

$$(7) \quad \alpha = \frac{1}{2^7 \cdot 3 \cdot \pi^2} \int_M (\|R\|^2 - 8\|S\|^2 + 3\rho^2) * 1,$$

$$(8) \quad \chi = \frac{1}{2^5 \cdot \pi^2} \int_M (\|R\|^2 - 4\|S\|^2 + \rho^2) * 1 .$$

If the Bochner curvature tensor vanishes, (1) implies $\|R\|^2 = 2\|S\|^2 - \frac{1}{6}\rho^2$. Thus (6), (7) and (8) reduce to

$$(9) \quad \tau = \frac{1}{2^5 \cdot 3^2 \cdot \pi^2} \int_M \rho^2 * 1 ,$$

$$(10) \quad \alpha = \frac{-1}{2^7 \cdot \pi^2} \int_M \|R\|^2 * 1 + \frac{7}{2^7 \cdot 3^2 \cdot \pi^2} \int_M \rho^2 * 1$$

$$(11) \quad \chi = \frac{-1}{2^5 \cdot \pi^2} \int_M \|R\|^2 * 1 + \frac{1}{2^4 \cdot 3 \cdot \pi^2} \int_M \rho^2 * 1 ,$$

respectively. From (9), (10) and (11) we obtain

$$(12) \quad 4\alpha - \chi = \tau \geq 0 .$$

This proves inequalities (i) and (ii). If equality of (i) or (ii) holds, then (2) and (3) imply

$$(13) \quad c_1^2 = 2c_2 .$$

By combing (13) with a theorem of Chen and Ogiue [4], we see that (M, g) is either flat or locally a product surface of two curves, one with constant positive Gauss curvature H , and the other with constant Gauss curvature $-H$. Conversely, if M is either flat or a locally product surface of this given type, then the Bochner curvature tensor vanishes and $\tau = 4\alpha - \chi = 0$.

The remaining part of this theorem follows immediately from Lemma 1 and (9), (10) and (11).

4. Some applications of Proposition 1

Let P^n be an n -dimensional projective space. An algebraic surface M is called a complete intersection surface if it can be holomorphically imbedded in P^n as the intersection of $n - 2$ nonsingular hypersurfaces M_1, \dots, M_{n-2} of P^n in general position, that is, the tangent spaces of M_1, \dots, M_{n-2} intersect transversally everywhere along the surface. In [5] Chen and Ogiue proved that the induced Kaehler metrics of complete intersection surfaces from the Fubini-Study metric on P^n are not Bochner-Kaehler metrics unless the complete intersection surface is a linear plane P^2 in P^n . The following theorem gives a generalization of Chen-Ogiue's result.

Theorem 1 [3]. *All complete intersection surfaces except P^2 admit no Bochner-Kaehler metric.*

Proof. Let M be a complete intersection surface which can be holomorphically imbedded in P^n as

$$M_1 \cap \cdots \cap M_{n-2},$$

where M_α , $\alpha = 1, \dots, n-2$, are nonsingular hypersurfaces of degree d_α respectively. Let \tilde{h} be the generator of $H^2(P^n, \mathbb{Z})$ corresponding to the divisor class of a hyperplane P^{n-1} . Then the total Chern class $c(P^n)$ of P^n is given by

$$(14) \quad c(P^n) = (1 + \tilde{h})^{n+1}.$$

Let $j: M \rightarrow P^n$ be the imbedding and ν be the normal bundle of $j(M)$ in P^n . Then the total Chern class of $c(\nu)$ of ν is given by

$$(15) \quad c(\nu) = (1 + d_1 h) \cdots (1 + d_{n-2} h),$$

where h is the image of \tilde{h} under the homomorphism

$$j^*: H^2(P^n, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}).$$

Since $j^*T(P^n) = T(M) \oplus \nu$ (Whitney sum), we find

$$j^*c(P^n) = c(M) \cdot c(\nu).$$

where $T(M)$ is the tangent bundle of M , and $c(M)$ is the total Chern class of M . Thus from (14) and (15) we get

$$(16) \quad (1 + h)^{n+1} = (1 + c_1 + c_2)(1 + d_1 h) \cdots (1 + d_{n-2} h),$$

which implies

$$(17) \quad c_1 = (n + 1 - \sum d_\alpha)h,$$

$$(18) \quad c_2 = \left\{ \frac{1}{2}n(n+1) - (n+1) \sum d_\alpha + \sum_{\alpha \leq \beta} d_\alpha d_\beta \right\} h^2.$$

Combining these two equations with the Hirzebruch signature theorem we may find that

$$(19) \quad \tau = \frac{1}{3}\{n + 1 - \sum d_\alpha^2\},$$

If the complete intersection surface M admits a Bochner-Kähler metric, then by Proposition 1 and (19) we find

$$\sum d_\alpha^2 \leq n + 1,$$

from which we may assume that $d_2 = \cdots = d_{n-2} = 1$, and d_1 is either 1 or 2. If d_1 is 1, M is a linear plane P^2 . If d_1 is 2, then M is a quadric in P^3 . Thus, by using (17) and (18) we find

$$\chi = 1 \quad \text{and} \quad \tau = 0 ,$$

contradicting Proposition 1.

Theorem 2. *Let M be a surface. Then any surface \bar{M} obtained from M by blowing up k points of M admits no Bochner-Kaehler metric whenever either*

$$k > \tau - \alpha \quad \text{or} \quad k > \frac{1}{4}(3\tau - \chi)$$

where τ, α and χ denote the Hirzebruch signature, the arithmetic genus and the Euler characteristic of M .

Proof. Since the arithmetic genus is a birational invariant, surfaces M and \bar{M} have the same arithmetic genus. On the other hand, topologically, blowing up a point on a surface is equivalent to attaching P^2 with opposite orientation which is denoted by \bar{P}^2 . Since \bar{M} is obtained from M by blowing up k points of M , \bar{M} is diffeomorphic to the direct sum $M \# k\bar{P}^2$. Since we have

$$\tau(M \# k\bar{P}^2) = \tau(M) - k ,$$

and

$$\chi(M \# k\bar{P}^2) = \chi(M) + k ,$$

this theorem then follows from Proposition 1.

A surface is called a rational surface if it is birationally equivalent to P^2 . For the later use we prove the following.

Lemma 2. *All rational surfaces except P^2 admit no Bochner-Kaehler metric.*

Proof. Since a rational surface is obtained from P^2 by blowing up and blowing down, and rational surface is diffeomorphic to either

$$S^2 \times S^2 \quad \text{or} \quad P^2 \# k\bar{P}^2 \quad (k \geq 0) .$$

Since $\chi(S^2 \times S^2) = 4$ and $\tau(S^2 \times S^2) = 0$. $S^2 \times S^2$ cannot admit Bochner-Kaehler metric by virtue of Proposition 1. If a surface $P^2 \# k\bar{P}^2$ admits a Bochner-Kaehler metric, then Proposition 1 implies that $k = 0$, because the arithmetic genus of all rational surface are equal to one. Thus by a result of Andreotti, the only Bochner-Kaehler metric on rational surfaces is the standard Fubini-Study metric on P^2 .

5. Some special surfaces

In the remaining part of this paper, we shall always assume that all surfaces are analytic.

In this section we shall introduce some special surfaces for later use. For the details, see for examples, Kodaira [7], [8], and Šafarevič [11], [12].

Let C_j be a line bundle or a divisor on M for $j = 1, 2$. We denote by $(C_1 C_2)$

the intersection number of C_1 and C_2 . By an *exceptional curve* (of the first kind) on a surface we mean a nonsingular connected rational curve such that the self-intersection number $(C^2) = -1$; it is known that a curve is exceptional if and only if it arises as the result of blowing up a point via a quadric transformation. A surface M is a *minimal surface* if it contains no exceptional curves.

A surface is said to be regular if $b_1 = 0$. A K3 surface is a regular surface with trivial canonical line bundle, i.e., $K = 0$, an Enriques surface is a regular surface with $2K = 0$. A Hirzebruch surface $F_n = P(H^n \oplus 1)$ is the projective bundle associated to the vector bundle $H^n \oplus 1$ over P^1 , H being the line bundle defined by a hyperplane section. A surface is of general type if we have

$$\overline{\lim} \frac{P_m}{m^2} > 0 .$$

It is known that all surfaces of general type are algebraic. By using the Atiyah-Singer index theorem, Van de Ven [13] proved that all such surfaces satisfying $c_1^2 \leq 8c_2$. Recently, Bogomulov (in Moscow) improved Van de Ven's result as follows: *All surfaces of general type satisfy $c_1^2 \leq 4c_2$* . Recently, Y. Miyako [9] shows that all surfaces of general type satisfy $c_1^2 \leq 3c_2$.

A surface is said to be *ruled* if it is birationally equivalent to the direct product of a curve with a projective line, a surface M is called an *elliptic surface* if there exists a curve C and a surjective morphism $f: M \rightarrow C$ such that a *generic fibre* of f is an algebraic curve of genus one (i.e., elliptic curve).

We consider an arbitrary regular mapping $\pi: V \rightarrow B$ of a surface V onto a nonsingular algebraic curve B with an irreducible generic fibre F . The fibre $F_b = \pi^{-1}(b)$ is connected for all $b \in B$. For all points $b \in B$, except perhaps, a finite number, F_b is an irreducible nonsingular algebraic curve with genus g . The set of points $\{b_1, \dots, b_s\}$ for which this is not true will be denoted by S , and the corresponding fibres F_{b_i} are called singular fibres. We state the following known lemmas for later use.

Lemma 3 [11, p. 58]. *Let $\chi(L)$ denote the Euler characteristic of a topological space L . Then*

$$\chi(V) = \chi(F)\chi(B) + \sum_{i=1}^s (\chi(F_{b_i}) - \chi(F)) .$$

Lemma 4 [11, p. 60]. *If F is nonsingular and F_0 is singular fibre of π and the surface V is minimal, then*

$$\chi(F_0) \geq \chi(F) ,$$

where equality holds only when the genus of F is equal to 1 and F_0 is a nonsingular curve of genus 1 taken with some multiplicity.

6. Classification of Bochner-Kaehler surfaces

In this section we shall give a classification of Bochner-Kaehler surfaces for the analytic case.

Theorem 3. *If an analytic surface M admits a Bochner-Kaehler metric g , then either (M, g) is a space form, or (M, g) is a locally product surfaces of two curves, one with constant positive Gauss curvature H , and the other with constant negative Gauss curvature $-H$.*

Proof. According to Kodaira's classification theorem [7, p. 796], [8, p. 1064], surfaces free from exceptional curves can be classified into the following seven classes:

- (I)₀ the class of projective plane and ruled surfaces,
- (II)₀ the class of K3 surfaces,
- (III)₀ the class of complex tori,
- (IV)₀ the class of minimal elliptic surfaces with $b_1 \equiv 0(2)$, $P_{12} > 0$, $K \neq 0$,
- (V)₀ the class of minimal algebraic surfaces with $P_2 > 0$, $c_1^2 > 0$,
- (VI)₀ the class of minimal elliptic surfaces with $b_1 \equiv 1(2)$, $P_{12} > 0$,
- (VII)₀ the class of minimal surfaces with $b_1 = 1$, $P_{12} = 0$.

Moreover, surfaces of classes (II)₀, (III)₀, (IV)₀ and (VI)₀ satisfy $c_1^2 = 0$, surfaces of class (II) satisfy $b_1 = 0$, and surfaces of class (III)₀ satisfy $b_1 = 4$.

In the following a surface is said to be of class (Y) if it is obtained from a minimal surfaces of class (Y)₀, (Y ranges from I to VII) by means of a finite number of quadric transformations. The corresponding minimal surfaces M_0 of M are called minimal models of M .

Since the first Betti number b_1 is a birational invariant and is even for Kaehler surfaces, which is equal to $2q$, all surfaces of classes (VI) and (VII) admit no Bochner-Kaehler metric. Now we consider the remaining cases separately.

(a) Surfaces of class (I). Surfaces in this class are either rational or ruled. If a surface is rational and admits a Bochner-Kaehler metric, then Lemma 2 implies that it is the projective plane and hence the Kaehler metric on it is the standard Fubini-Study metric. So we may assume that the surface M is ruled and nonrational. Thus the surface M is birationally equivalent to the direct product of the projective line P^1 with a curve C of genus $g \geq 1$.

Case (i). If the genus of C is one, M is elliptic. Hence a minimal model M_0 of M satisfies

$$(20) \quad \chi(M_0) \geq 0, \quad c_1^2(M_0) = 0$$

by virtue of Lemmas 3 and 4 and Theorem 3 of [11, p. 166]. On the other hand, since the arithmetic genus α of M is equal to zero, Proposition 1 shows that M admits no Bochner-Kaehler metric unless $\chi(M) \leq 0$. Since blowing up a point will increase the Euler characteristic by one, (20) shows that M admits no Bochner-Kaehler metric unless $\chi(M) = 0$, and M itself is a minimal surface. From this we find that if M admits a Bochner-Kaehler metric, then $c_1^2 = \chi =$

$\tau = 0$. Applying Proposition 1 again, we see that M is a flat surface.

Case (ii). If genus g of C is ≥ 2 , then

$$(21) \quad \alpha = 1 - g \leq -1 .$$

Let M_0 be the minimal model of M by blowing down all exceptional curves of M . Then Lemmas 3 and 4 imply that

$$(22) \quad \chi(M_0) \geq 4(1 - g) .$$

Since M is obtained from M_0 by blowing up, we find

$$(23) \quad \chi(M) \geq 4(1 - g) .$$

Thus, if M admits a Bochner-Kaehler metric, then (21), (23) together with Proposition 1 imply that (a) M is a minimal surface, (b) all fibres are generic, and (c) $\chi(M) = 4(1 - g)$. From these we may conclude that M is the direct product of P^1 with a curve of genus $g \geq 2$ and $\tau = 0$. Hence by applying Proposition 1 again, all Bochner-Kaehler metric is the given locally product one.

(b) Surfaces of class (II). All surfaces in this class are regular. Thus their Euler characteristics satisfy

$$(24) \quad \chi \geq 3 .$$

On the other hand, since $c_1^2 = 0$ for minimal $K3$ surfaces we get

$$(25) \quad c_1^2 \leq 0$$

for every surface M in this class. From these it follows that the Hirzebruch signature of every M satisfies

$$(26) \quad \tau \leq -2 .$$

This together with Proposition 1 shows that all surfaces in this class admit no Bochner-Kaehler metric.

(c) Surfaces of class (III). Surfaces in this class satisfy [7, p. 796]

$$(27) \quad b_1 = 4 , \quad p_g = 1 .$$

Thus we get

$$(28) \quad \alpha = 1 - q + p_g = 0 .$$

On the other hand, since the minimal models M_0 of these surfaces satisfy

$$(29) \quad \chi(M_0) = 0 , \quad c_1^2(M_0) = 0 ,$$

we find

$$(30) \quad \chi(M) \geq 0,$$

for all surfaces M in this class. Thus using index theorems again we find

$$\tau = 4\alpha - \chi \leq 0,$$

where the equality holds only when the surfaces M are minimal. By Proposition 1 we see that all surfaces in this class admit no Bochner-Kaehler metric unless they are minimal in which case we have

$$\alpha = \tau = \chi = 0.$$

Thus the Bochner-Kaehler metrics are the flat ones.

(d) Surfaces of class (IV). Surfaces in this class are elliptic surfaces. Thus minimal models M_0 of these surfaces satisfy [11, p. 166]

$$(31) \quad c_1^2(M_0) = 0.$$

On the other hand, Lemmas 3 and 4 imply that $\chi(M_0) \geq 0$ which together with (31) yields

$$\tau(M) = \frac{1}{3}(c_1^2(M) - 2c_2(M)) \leq 0.$$

Combining this with Proposition 1 we see that if a surface M admits a Bochner-Kaehler metric, then M is minimal and the metric is the flat one.

(e) Surfaces of class (V). Surfaces in this class are algebraic surfaces of general type. Combining Proposition 1, Miyaoka's theorem and Theorem 3 of [4] we see that if a surface M admits a Bochner-Kaehler metric g , then we have either

$$(32) \quad \tau = 0 \text{ (i.e., } c_1^2 = 2c_2) \text{ or } c_1^2 = 3c_2,$$

The first case holds only when (M, g) is either flat or the locally product surface of the given type. The second case implies that (M, g) is a (complex) space form [4].

As an immediate consequence of Theorem 3, we have

Theorem 4. *An analytic surface admits a Bochner-Kaehler metric if and only if it is covered biholomorphically by P^2 , C^2 , D^2 or $P^1 \times D^1$, where D^i is the disc of C^i .*

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